KZG Polynomial Commitment Scheme on zk-SNARKs Construction and Its Implementation

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Final Project Defense
June 2025

Outline

What are Zero-Knowledge Proofs?

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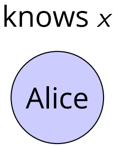

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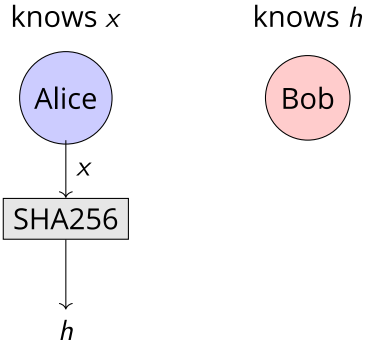

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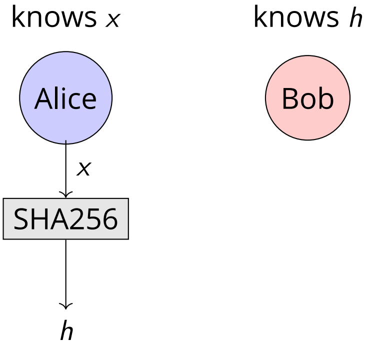

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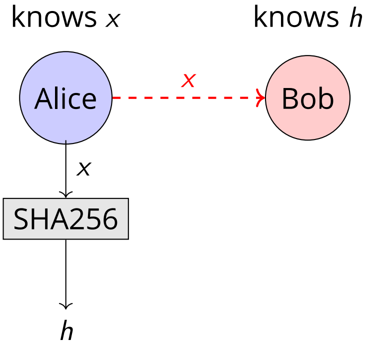

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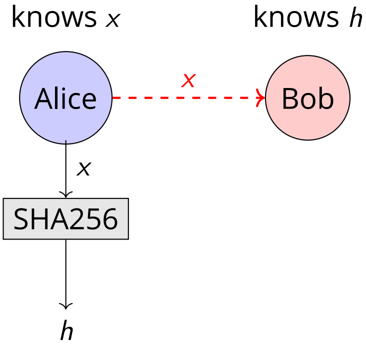

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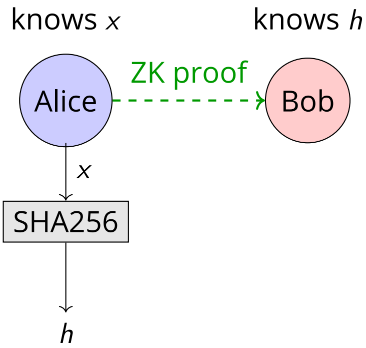

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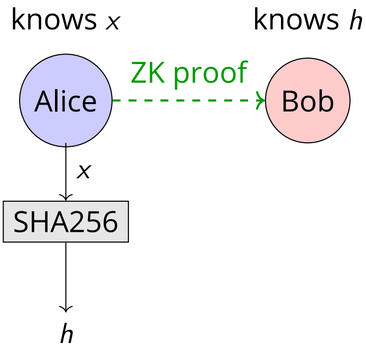
With Zero-Knowledge:

- ✓ Alice proves knowledge of x
- ✓ Bob learns nothing about x



The Educational Challenge

High-level overviews

Lack mathematical rigor

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Research papers

Dense notation

Assumed expertise

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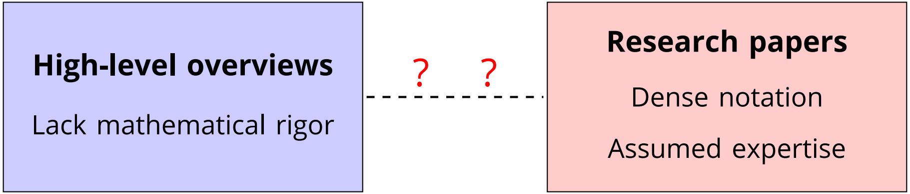

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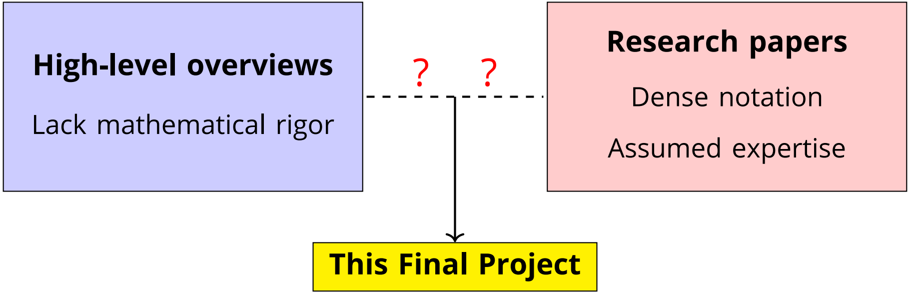

Accessible exposition + Concrete examples + Working implementations

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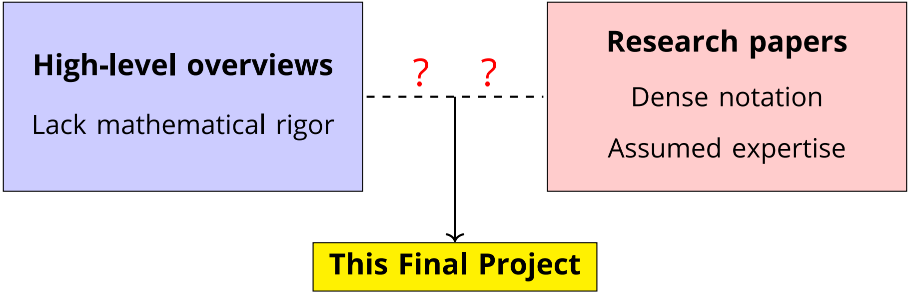

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Why this matters: Growing importance in blockchain, privacy technologies, and secure computation

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Core Focus

KZG polynomial commitment scheme and its application in two prominent zk-SNARKs protocols: Marlin and Plonk

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- **N**on-interactive: No back-and-forth communication needed
- **AR**gument of **K**nowledge: Prover must know the witness

The zk-SNARKs Ecosystem

Framework	Frontend	Language	Proof System
Arkworks	Self-contained	Rust	Groth16, Marlin, GM17, Plonk
Gnark	Self-contained	Go	Groth16, Plonk (KZG, FRI)
Hyrax	None	Python	Hyrax
LEGOSnark	None	C++	Brakedown-like
LibSNARK	xJsnark	Java, C++	Groth16, Pinocchio, GGPR
Zokrates	Self-contained	Zokrates DSL	Groth16, GM17, Marlin, Nova
Mirage	None	Java	Pinocchio-like
PySNARK	Self-contained	Python	Groth16
SnarkJS	Circom	JavaScript, Circom DSL	Groth16, Plonk (via WASM)
Rapidsnark	Circom	JavaScript, Circom DSL	Groth16
Spartan	None	Rust	Spartan
Aurora (libiop)	None	C++	Aurora
Fractal (libiop)	None	C++	Fractal
Virgo	None	Python	Virgo
Noir	Self-contained	Rust (Noir DSL)	Any ACIR-compatible system
Dusk-PLONK	None	Rust	PLONK
Halo2	None (Rust API)	Rust	PLONK-like

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SnarkJS	Circom	JavaScript, Circom DSL	Groth16, Plonk (via WASM)
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Adapted from “Zero-Knowledge Proof Frameworks: A Survey” by Sheybani et al. (2025)

Performance Comparison

Protocol	Proof Size	Prover	Verifier	Setup
Groth16	$2\mathbb{G}_1 + 1\mathbb{G}_2$	$O(n \log n)$	$O(\lvert \mathbb{x} \rvert)$	Circuit-specific
Marlin	$8\mathbb{F}_q + 13\mathbb{G}_1$	$O(n \log n)$	$O(\lvert \mathbb{x} \rvert + \log n)$	Universal
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Key Insight

Trade small efficiency loss for huge flexibility gain!

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Setup 1
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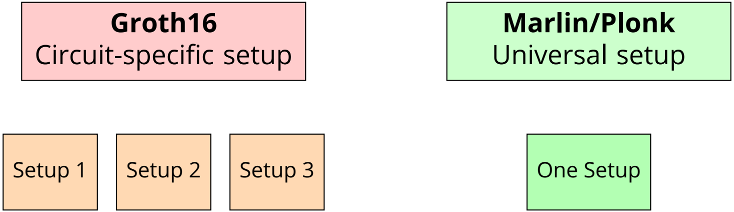

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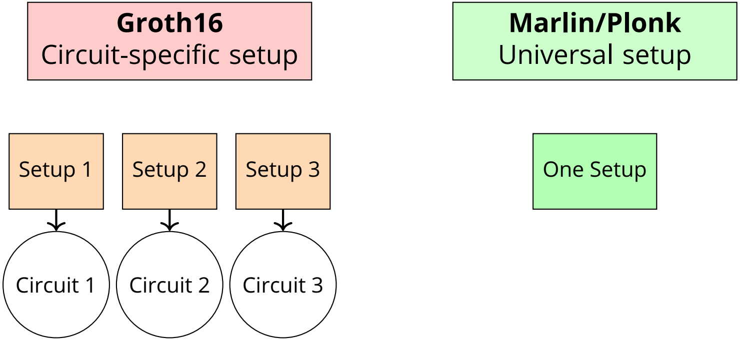

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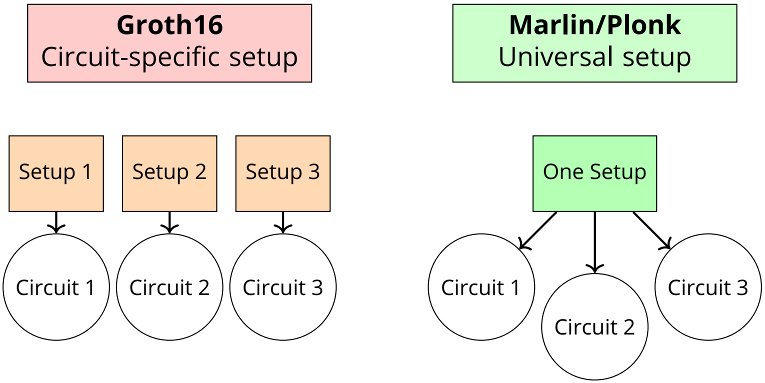

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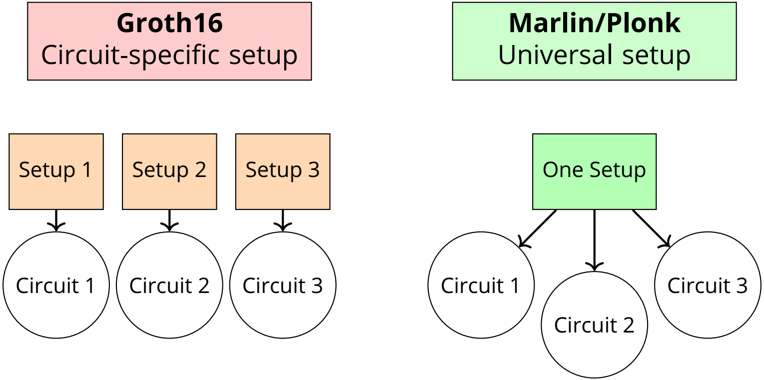

Key Advantage

Enabled by: KZG polynomial commitment scheme with updatable SRS!

Mathematical Foundations

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Building blocks for modern cryptography

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In cryptography: $q \approx 2^{256}$ for security

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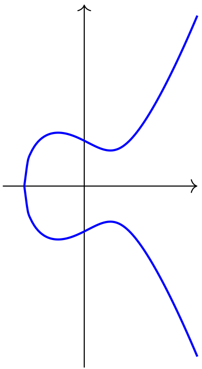

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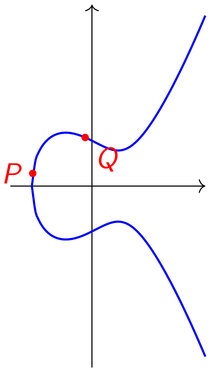

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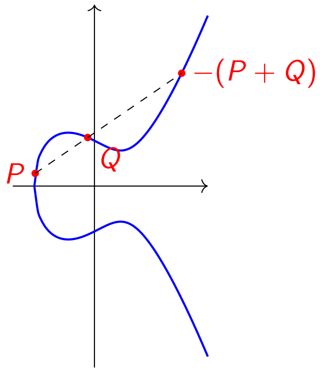

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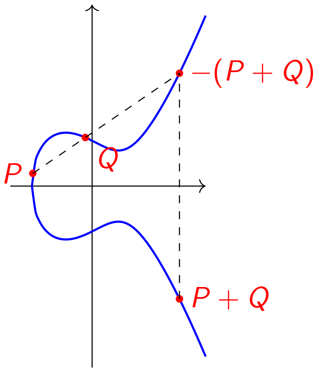

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- Examples: Weil pairing, Tate pairing

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(1) A functional commitment scheme (cryptographic object)

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(2) A compatible interactive proof (info. theoretic object)

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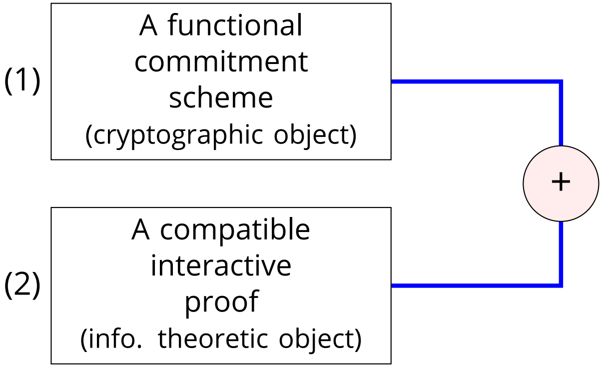

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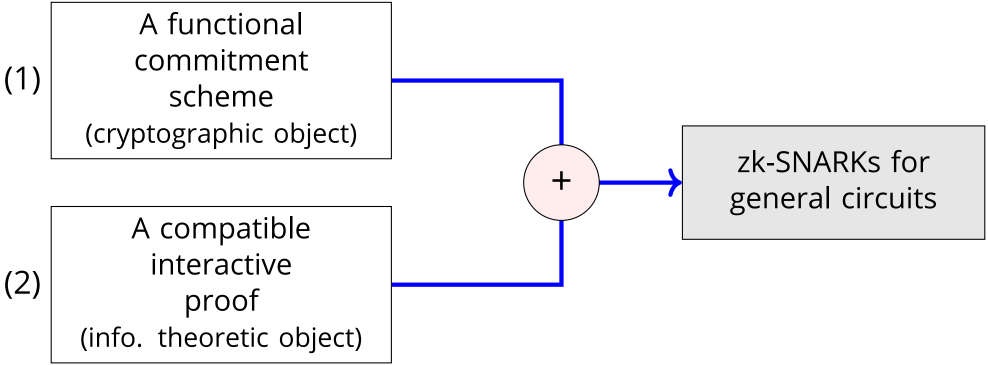

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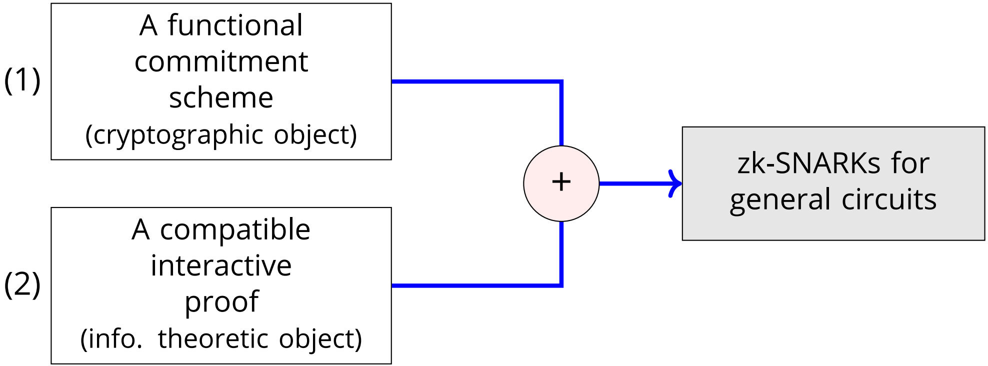

In this work

KZG (same for both) + Different interactive proofs (Marlin vs Plonk)

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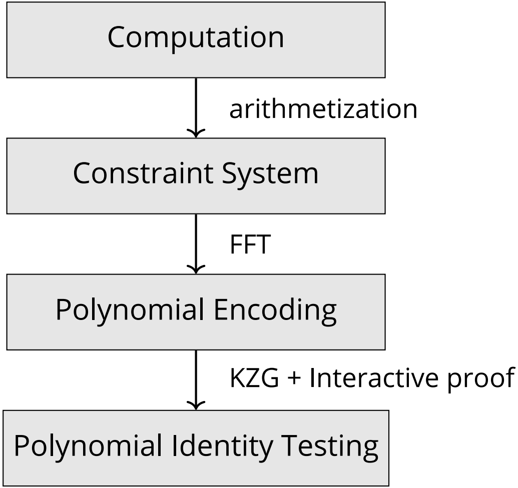
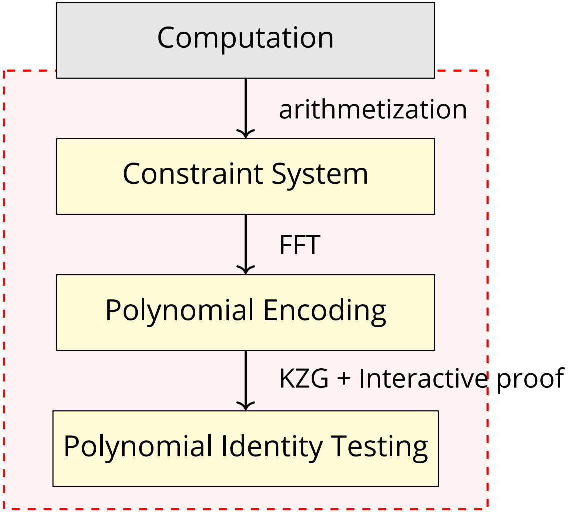
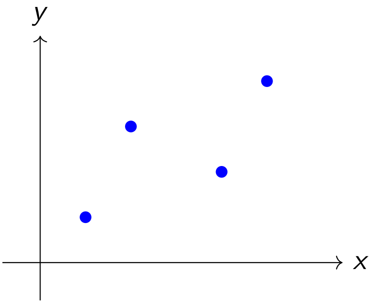


The Computation Pipeline

Scope of this final project



Why Polynomials?



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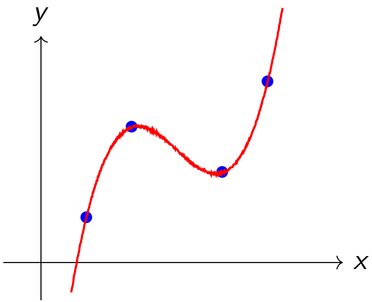

Polynomial through points

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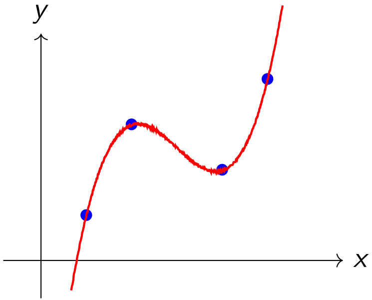

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Unique Interpolation

- n points uniquely determine degree $n - 1$ polynomial

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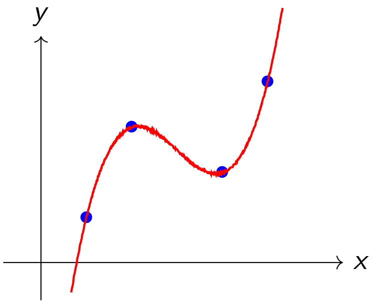

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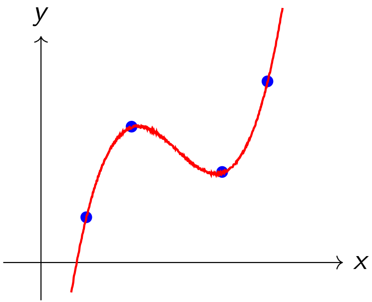

Polynomial through points

Unique Interpolation

- n points uniquely determine degree $n-1$ polynomial
- Can encode n values as single object
- Efficient algorithms (FFT) with $O(n \log n)$

The Schwartz-Zippel Lemma

Lemma

Let $\mathbb{F}_q$ be a finite field and $f \in \mathbb{F}_q[X_1, X_2, \ldots, X_n]$ be a non-zero polynomial of total degree at most d. Then

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In practice: $q \approx 2^{256}$, so probability $\leq \frac{d}{2^{256}}$ is negligible!

Polynomial Identity Testing Protocol

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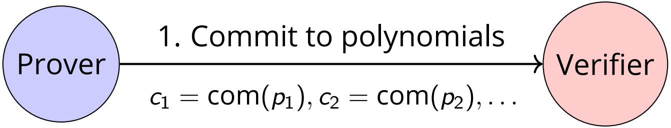

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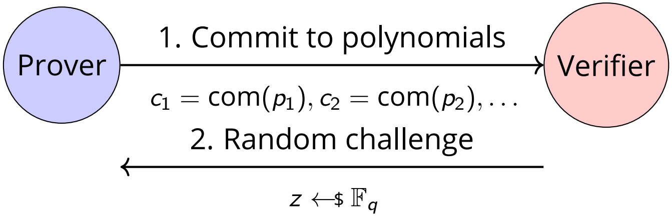

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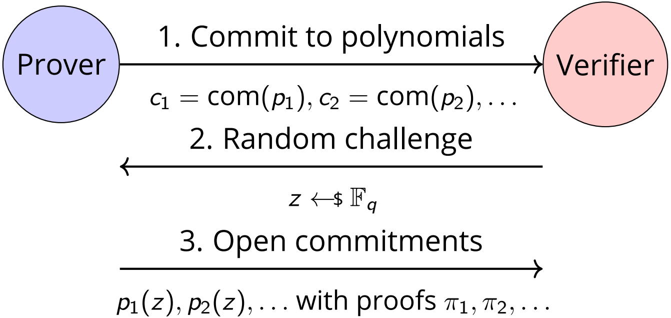

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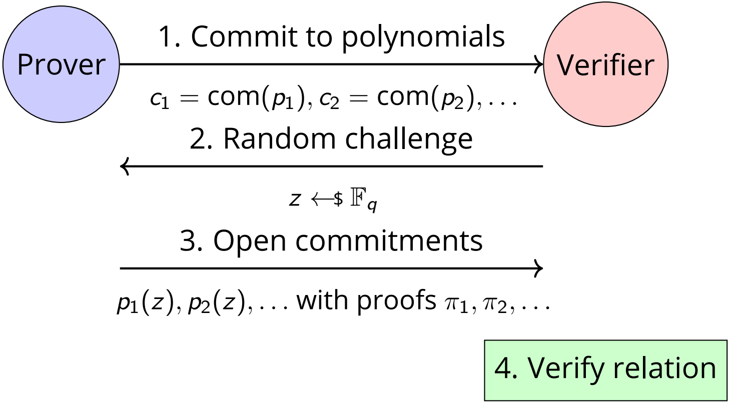

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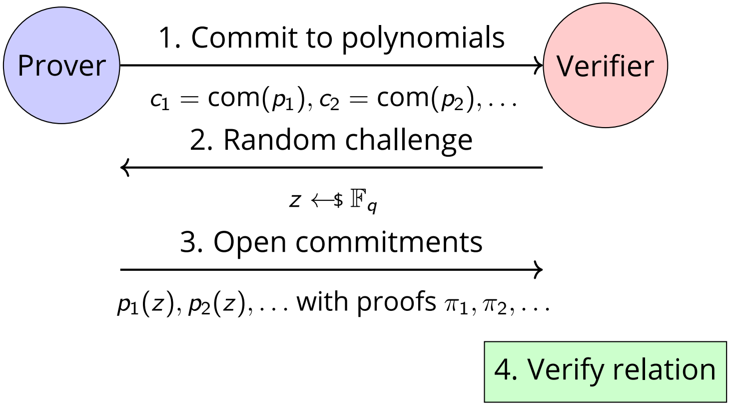

Security guarantees:

- **Binding**: Cannot change polynomials after commitment

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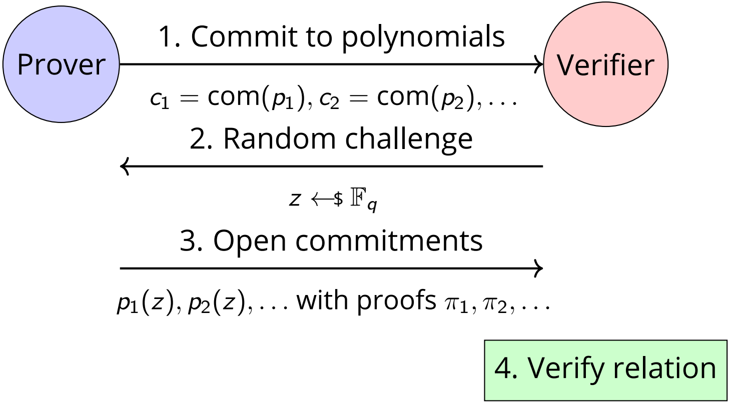
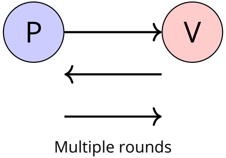


Security guarantees:

- **Binding**: Cannot change polynomials after commitment
- **Soundness**: Schwartz-Zippel ensures false claims fail with probability $\geq 1 - \frac{d}{|\mathbb{F}_q|}$

Interactive to Non-Interactive

Interactive Protocol



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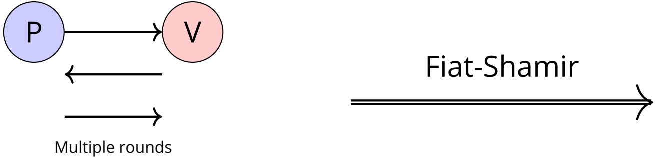

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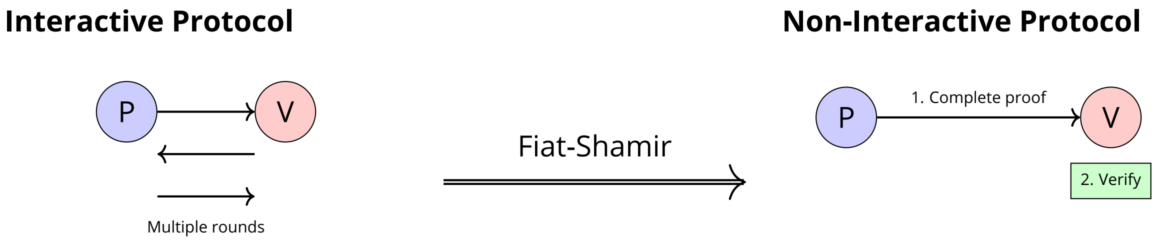
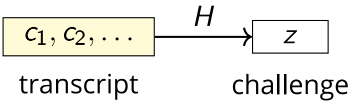


Interactive to Non-Interactive

Interactive Protocol **Non-Interactive Protocol**

Key transformation: Replace verifier's random challenges with hash function



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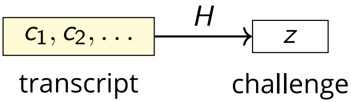

Result

Single proof string that can be verified by anyone - perfect for blockchain!

KZG Construction Overview

Setup Phase (Trusted)

Generate powers of secret x in $\mathbb{F}_q$:

$$\text{SRS} = \{G_1, xG_1, x^2 G_1, \ldots, x^d G_1, G_2, xG_2\}$$

Secret x is destroyed after setup!

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3. **Verify**: Check using bilinear pairing

Committing to Polynomials

Example: $p(X) = 2X^2 + 3X + 5$

a_0	5
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a_2	2

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Using SRS powers:

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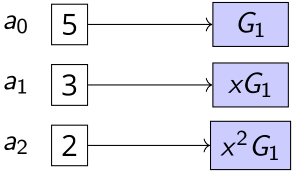

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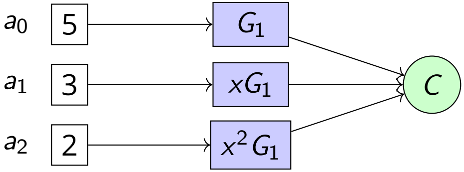

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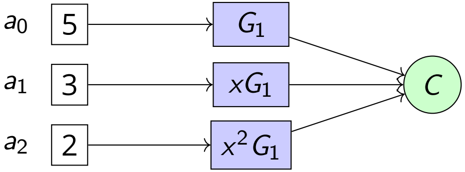

Result: Single group element C!

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Theorem

For $p \in \mathbb{F}_q[X]$ and $z, v \in \mathbb{F}_q$,

$$p(z) = v \iff (X - z) \text{ divides } (p(X) - v)$$

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Example: If $p(X) = X^2 + 2X + 1$ and claiming $p(3) = 16$:

$$w(X) = \frac{X^2 + 2X + 1 - 16}{X - 3} = \frac{X^2 + 2X - 15}{X - 3} = X + 5$$

Verification via Pairing

Verification Equation

$$e(C - vG_1, G_2) \stackrel{?}{=} e(\pi, xG_2 - zG_2)$$

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Why this works:

$$\text{LHS} = e((p(x) - v)G_1, G_2) = e(w(x)(x - z)G_1, G_2) = e(w(x)G_1, (x - z)G_2) = \text{RHS}$$

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- Constant time verification!

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Remark

Complete zk-SNARKs actually need a stronger property than evaluation binding: **extractability**. This ensures any valid commitment corresponds to an actual polynomial (as required in the Marlin paper).

Proving Evaluation Binding

Strong Diffie-Hellman (SDH) Assumption

Given $\{G_1, xG_1, x^2G_1, \ldots, x^dG_1, G_2, xG_2\}$, hard to compute:

$$\left(c, \frac{1}{x+c}G_1\right) \text{ for any } c \in \mathbb{F}_q$$

Proof idea: If adversary breaks binding $\Rightarrow$ can break SDH
Suppose adversary outputs (C, z, v, v', π, π') with $v \neq v'$
Both proofs verify:

$$e(C - vG_1, G_2) = e(\pi, xG_2 - zG_2)$$
$$e(C - v'G_1, G_2) = e(\pi', xG_2 - zG_2)$$

Subtracting:

$$e((v' - v)G_1, G_2) = e(\pi - \pi', xG_2 - zG_2)$$

If $\pi \neq \pi'$**:** Can extract $\frac{1}{x-z}G_1 = \frac{\pi-\pi'}{v'-v} \Rightarrow$ Breaks SDH!

Constraint System

Marlin (R1CS)

Constraint equation:

$$\boxed{Az \circ Bz = Cz}$$

Where:

- $A, B, C \in \mathbb{F}_q^{n \times n}$ are constraint matrices
- $z = (x, w) \in \mathbb{F}_q^n$ is the assignment vector
- x are public inputs, w are witness values
- $\circ$ denotes entry-wise product

Plonk

Gate constraint:

$$\boxed{\begin{aligned} q_L \cdot z_{a_i} + q_R \cdot z_{b_i} + q_O \cdot z_{c_i} \\ + q_M \cdot (z_{a_i} \cdot z_{b_i}) + q_C = 0 \end{aligned}}$$

Where:

- $q_L, q_R, q_O, q_M, q_C \in \mathbb{F}_q^n$ are selectors
- $z = (x, w) \in \mathbb{F}_q^m$ is wire assignment
- x are public inputs, w are witness values
- $a, b, c \in [m]^n$ are wire indices
- Additional copy constraints via σ

Example Problem Definition

Polynomial Evaluation Problem

Prove knowledge of secret $X \in \mathbb{F}_{23}$ such that:

$$Y = X^3 + 2X + 5$$

where $Y = 15$ is public and $X = 3$ is the witness.

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- $w_2 = X^2 = 9$
- $w_3 = X^3 = 4$ (note: $27 \bmod 23 = 4$)

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- $w_1 = X = 3$
- $w_2 = X^2 = 9$
- $w_3 = X^3 = 4$ (note: $27 \bmod 23 = 4$)
- $w_4 = 2X = 6$

Example Problem Definition

Polynomial Evaluation Problem

Prove knowledge of secret $X \in \mathbb{F}_{23}$ such that:

$$Y = X^3 + 2X + 5$$

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Marlin R1CS Encoding

Assignment vector: $z = [1, 15, 3, 9, 4, 6]$

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3. $2 \cdot w_1 = w_4$ (computing $2X$)
4. $(5 + w_3 + w_4) \cdot 1 = Y$ (final addition)

Marlin Constraint Matrices

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note: Matrices are 4×6 (4 constraints, 6 variables). In practice, padded with zero rows to form square $n \times n$ matrices for polynomial encoding.

Plonk Circuit Encoding

Wire assignment: $z = [1, 2, 5, 15, 3, 9, 4, 6, 10]$

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1-4. Constant gates: $z_1 = 1, z_2 = 2, z_3 = 5, z_4 = 15$

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6. $z_6 \cdot z_5 = z_7$ (computing X^3)

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8. $z_7 + z_8 = z_9$ (computing $X^3 + 2X$)
9. $z_9 + z_3 = z_4$ (final addition)

Plonk Selector Vectors and Wire Indices

Selector vectors and wire indices:

	G1	G2	G3	G4	G5	G6	G7	G8	G9
q_L	1	1	1	1	0	0	0	1	1
q_R	0	0	0	0	0	0	0	1	1
q_O	0	0	0	0	-1	-1	-1	-1	-1
q_M	0	0	0	0	1	1	1	0	0
q_C	-1	-2	-5	-15	0	0	0	0	0
a	1	2	3	4	5	6	5	7	9
b	0	0	0	0	5	5	2	8	3
c	0	0	0	0	6	7	8	9	4

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	G1	G2	G3	G4	G5	G6	G7	G8	G9
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q_M	0	0	0	0	1	1	1	0	0
q_C	-1	-2	-5	-15	0	0	0	0	0
a	1	2	3	4	5	6	5	7	9
b	0	0	0	0	5	5	2	8	3
c	0	0	0	0	6	7	8	9	4

Copy constraints: Permutation σ ensures wire consistency $\sigma = (1)(2,16)(3,18)(4,27)(5,7,14,15)(6,23)(8,24)(9,26)(10,11,12,13,19,20,21,22)(17,25)$

Polynomial Encoding

Marlin **Plonk**

Polynomial Encoding

Marlin

Witness polynomials:

- $\hat{w}(X)$ - shifted witness

Plonk

Marlin

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- $\hat{z}(X)$ - full assignment

Plonk

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- $\hat{z}_A(X), \hat{z}_B(X), \hat{z}_C(X)$ - linear combinations

Plonk

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Wire polynomials:

- $a(X), b(X), c(X)$ - left, right, output

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Selector polynomials:

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Permutation polynomials:

- $S_{\sigma_1}(X), S_{\sigma_2}(X), S_{\sigma_3}(X)$

Polynomial Identity Testing

Marlin

Entry-wise product constraint:

$$\hat{z}_A(X)\hat{z}_B(X) - \hat{z}_C(X) = h_0(X)v_H(X)$$

First sumcheck relation:

$$s(X) + r(\alpha, X)\sum_M \eta_M \hat{z}_M(X) - t(X)\hat{z}(X) = h_1(X)v_H(X) + Xg_1(X)$$

Second sumcheck relation:

$$a(X) - b(X)q_2(X) = h_2(X)v_K(X)$$

Plonk

Gate constraint:

$$q_L(X)a(X) + q_R(X)b(X) + q_O(X)c(X) + q_M(X)a(X)b(X) + q_C(X) + (X) = h_0(X)v_H(X)$$

Permutation first:

$$L_1(X)(Z(X) - 1) = q_1(X)v_H(X)$$

Permutation second:

$$Z(X)f'(X) - g'(X)Z(X) = q_2(X)v_H(X)$$

Marlin vs Plonk: Performance Comparison

Metric	Marlin	Plonk
Constraint System	R1CS	Custom gates
SRS degree	$6m$	n
Proof size ($\mathbb{F}_q$)	8	**6**
Proof size ($\mathbb{G}_1$)	13	**9**
Prover v-MSM operations	11	**7**
Verifier field operations	$O(\ell + \log m)$	$O(\ell + \log n)$

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Both achieve universal & updatable SRS via KZG!

Implementation Architecture

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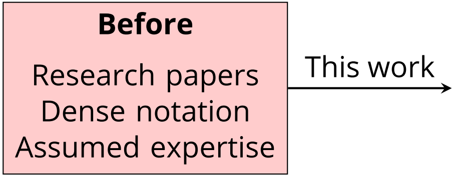
Educational Impact

Before

Research papers
Dense notation
Assumed expertise

Educational Impact



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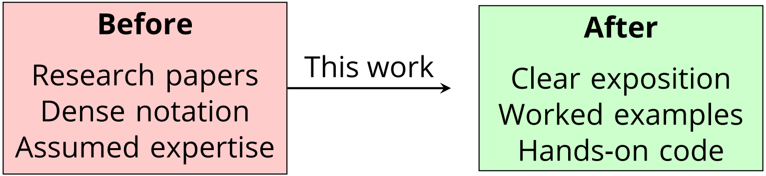
Hands-on code

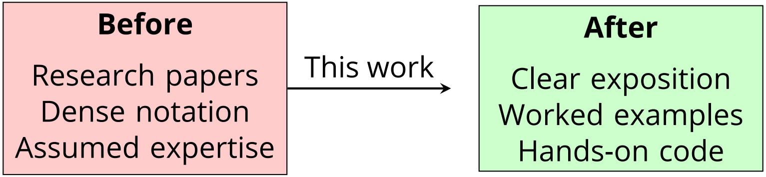
Educational Impact



Enables students to:

- Understand core concept

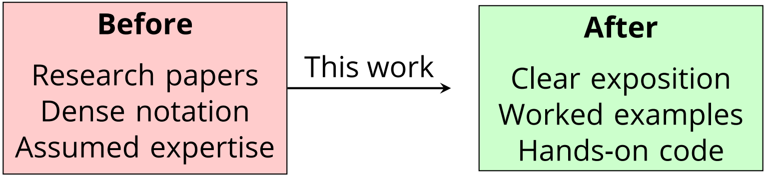
Educational Impact



Enables students to:

- Understand core concept
- Experiment with parameters and see effects

Educational Impact



Enables students to:

- Understand core concept
- Experiment with parameters and see effects
- Build foundation for advanced study in ZKPs

Thank You!

Questions?

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